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# The 'closed subgroup theorem' for localic herds and pregroupoids

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*Abstract*

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In this paper we extend the result that a localic subgroup of a localic group is necessarily closed, and the 'fibrewise' generalization of this result to localic groupoids, to algebras in the category of locales for more general algebraic theories such as quasigroups and herds (associative Mal'cev algebras), and to pregroupoids in the sense of A. Kock.

## Introduction

The theorem that every localic subgroup of a localic group is closed was first proved in [5]. Subsequently, a simpler proof was given in [7]; this proof was 'constructivized' and extended to localic groupoids in [8], by introducing the concept of fibrewise closedness. Our concern in the first part of this paper is to extend the result in a different direction, by analysing how much of the algebraic structure of groups we actually need to use in proving the theorem; we shall show that it remains valid for a class of 'generalized quasigroups' which includes both quasigroups in the usual sense [4] and associative Mal'cev algebras (which, following Lambek [12], we call *herds*). Whether the theorem remains valid for the still larger class of all Mal'cev algebras is still an open question, but it seems unlikely. In the second half of the paper, we prove the theorem for a common generalization of herds and groupoids, equivalent to the *pregroupoids* of Kock [10]. No doubt one could also formulate the theorem for a common generalization of groupoids and our generalized quasigroups; we have not done so, largely because of the lack of any motivating examples.

## 1. Quasigroups and herds

We begin by noting that the 'closed subgroup theorem' does not extend to localic semigroups, or even to monoids (semigroups with 1): the set $\{0\} \cup (1, \infty)$ is a non-closed submonoid of the additive group of reals, and its local compactness ensures that it yields a counterexample in localic monoids as well as topological monoids (cf. [6, II 2.13]). Thus the existence of inverses, in some form, must play an essential role in the proof of the theorem. On the other hand, the identity element does not play an important role in the proof as given in [8] (despite its prominent appearance in 'Wraith's Lemma' in [7]); nor does the associativity of the multiplication. One might, therefore, expect that the natural setting for the proof would be the variety of quasigroups, which differ from groups precisely in lacking the two latter features. However, with the class of herds also in mind, we have chosen to formulate the theorem in a more general variety, which does not seem to have been much studied before, and whose members we have christened *ternary quasigroups*.

A ternary quasigroup is a set $A$ equipped with three ternary operations $\mu$, $\lambda$ and $\varrho$ satisfying the four identities

$$\lambda(x, y, \mu(x, y, z)) = z, \qquad \mu(x, y, \lambda(x, y, z)) = z,$$

$$\varrho(\mu(x, y, z), y, z) = x, \qquad \mu(\varrho(x, y, z), y, z) = x.$$

(Thus if we think of $\mu$ as 'multiplication', $\lambda(x, y, -)$ is 'left division by $(x, y)$', and $\varrho$ is similarly interpreted as 'right division'.) An ordinary (binary) quasigroup may be regarded as a special case of a ternary quasigroup in which the three operations $\mu$, $\lambda$ and $\varrho$ happen not to depend on their middle variable $y$ (and so may be written as binary operations). Conversely, if $a$ is any element of a ternary quasigroup $A$, then we get a binary quasigroup structure on $A$ by 'fixing the middle variable at $a$' (i.e. by defining $\mu'(x, y) = \mu(x, a, y)$, etc.). However, the latter process is an 'unnatural' one: a homomorphism $f: A \to B$ of ternary quasigroups will not in general respect the chosen elements of $A$ and $B$, and so will not be a homomorphism of binary quasigroups. Also, it is perfectly possible (we shall see an example later) to have a localic ternary quasigroup which is nontrivial but does not have any points, and so is not reducible to a binary quasigroup in this way.

One could also consider $n$-ary quasigroups for $n > 3$, but there is no extra generality in this: given three $n$-ary operations $\mu$, $\lambda$, $\varrho$ on a set $A$ satisfying the $n$-ary analogues of the equations above, we obtain a ternary quasigroup structure by setting

$$\mu'(x, y, z) = \mu(x, y, y, \ldots, y, z)$$

(where the variable $y$ is repeated $n-2$ times), etc. Moreover, the process of setting variables equal, unlike that of choosing a constant, is natural—that is, it defines a (faithful) functor from the category of $n$-ary quasigroups to that of ternary quasigroups.

We may now state the main result of this section. Note that here, and subsequently in this section, words and phrases enclosed in square brackets may be ignored if one is working non-constructively, as in [7], but must be taken into account in a constructive context, as in [8].

**Theorem 1.1.** *Let $G$ be a localic ternary quasigroup, and $H$ a sublocale of $G$ which is a sub-ternary-quasigroup* [*and such that the unique locale map $H \to 1$ is open*]. *Then $H$ is* [*weakly*] *closed in $G$.*

**Proof.** *Let $\bar{H}$* denote the [weak] closure of $H$ in $G$. By [8, Corollary 1.13], $\bar{H}$ is also a sub-ternary-quasigroup of $G$, so we may reduce to the case when $\bar{H}=G$, i.e. when $H$ is [strongly] dense in $G$. [Moreover, $\bar{H} \to 1$ is open by [8, Lemma 1.11(ii)].] So it suffices to prove the following:

**Lemma 1.2.** *Let $G$ be a localic ternary quasigroup* [*such that $G \to 1$ is open*], *and let $H$ be a* [*strongly*] *dense sublocale of $G$ which is a sub-ternary-quasigroup. Then $H$ is the whole of $G$.*

Lemma 1.2 in turn follows immediately on setting $S=T=U=H$ in the following:

**Lemma 1.3.** *Let $G$ be a localic ternary quasigroup* [*such that $G \to 1$ is open*], *and let $S$, $T$, $U$ be any three* [*strongly*] *dense sublocales of $G$. Then*

(i) *The inclusion $S \times T \times U \to G \times G \times G$ is fibrewise dense over $\mu : G \times G \times G \to G$.*

(ii) *The composite $S \times T \times U \to G \times G \times G \xrightarrow{\mu} G$ is epimorphic.*

**Proof.** The second assertion follows immediately from the first and [8, Lemma 1.11(i)], since either the second or the fourth of the equations defining a ternary quasigroup implies that $\mu : G^3 \to G$ is (split) epimorphic. Now by [8, Proposition 1.12], we know that $S \times T$ is [strongly] dense in $G \times G$; and we have a diagram

$$\begin{tikzcd}
S \times T \times G \arrow[r] \arrow[d, "(\pi_1, \pi_2)"] & G \times G \times G \arrow[r, "\pi_3"] \arrow[d, "(\pi_1, \pi_2)"] & G \arrow[d] \\
S \times T \arrow[r] & G \times G \arrow[r] & 1
\end{tikzcd}$$

in which the squares are pullbacks. So by [8, Proposition 2.7], the inclusion $S \times T \times G \to G \times G \times G$ is fibrewise dense over $\pi_3$. But we also have a commutative diagram

$$\begin{tikzcd}
S \times T \times G \arrow[rr, "(\pi_1, \pi_2, \lambda')"] \arrow[d] & & S \times T \times G \arrow[d] \\
G \times G \times G \arrow[rr, "(\pi_1, \pi_2, \lambda)"] \arrow[dr, "\pi_3"'] & & G \times G \times G \arrow[dl, "\mu"] \\
& G &
\end{tikzcd}$$

in which the horizontal maps are isomorphisms ($\lambda'$ is the restriction of $\lambda$ to $S\times T\times G$), so $S\times T\times G\to G\times G\times G$ is fibrewise dense over $\mu$. A similar argument with $\varrho$ in place of $\lambda$ shows that $G\times T\times U\to G\times G\times G$ is fibrewise dense over $\mu$; so $S\times T\times U$, being the intersection of these two sublocales, is fibrewise dense over $\mu$ by [8, Lemma 1.2]. This completes the proof of Lemma 1.3, and hence of Theorem 1.1. $\square$

We now turn to herds. Recall that a ternary operation $\mu$ is called a *Mal'cev operation* [14] if it satisfies the identities $\mu(x,y,y)=x$ and $\mu(x,x,y)=y$. A Mal'cev operation $\mu$ is said to be *associative* if it also satisfies

$$\mu(x,y,\mu(z,u,v)) = \mu(\mu(x,y,z),u,v).$$

A set equipped with an associative Mal'cev operation is called a *herd*. (The original German term [13, 1] is "Schar"; Bruck [2] translates this as "flock", but we prefer "herd", which was apparently first used by Lambek [12].) To show that Theorem 1.1 applies to localic herds, we need the following:

**Lemma 1.4.** *A herd has a natural ternary quasigroup structure.*

**Proof.** Define $\lambda(x,y,z)=\mu(y,x,z)$ and $\varrho(x,y,z)=\mu(x,z,y)$. Then it is easy to verify that the ternary quasigroup identities follow from the herd identities: for example

$$\begin{aligned}\lambda(x,y,\mu(x,y,z)) &= \mu(y,x,\mu(x,y,z))\\ &= \mu(\mu(y,x,x),y,z)\\ &= \mu(y,y,z) = z. \qquad \square\end{aligned}$$

In the converse direction, ternary quasigroups form a Mal'cev variety [14]: given $\mu$, $\lambda$ and $\varrho$, we may define a Mal'cev operation $\bar{\mu}$ by

$$\bar{\mu}(x,y,z) = \mu(\varrho(x,t,\lambda(u,t,y)),t,\lambda(u,t,z)),$$

where $t$ and $u$ may (independently) be taken to be any of the three variables, $x$, $y$ and $z$, or indeed any term in these three variables. If the ternary quasigroup structure derives from a herd structure as in Lemma 1.4, then $\bar{\mu}(x,y,z)$ reduces to $\mu(x,y,z)$ (for any choice of $t$ and $u$), but in general it will be different, and there is no reason why it should be associative.

It is well known that a group becomes a herd if we define $\mu(x,y,z)=xy^{-1}z$. Conversely, given a nonempty herd $G$ and a choice of an element $e\in G$, we obtain a group structure (with $e$ as identity element) by setting $xy=\mu(x,e,y)$ and $x^{-1}=\mu(e,x,e)$ [3]. However, the latter construction is 'unnatural' in the sense that it fails to respect herd homomorphisms: for example, the nonempty subherds of a group are not just its subgroups but all cosets of its subgroups [1]. (Thus the relationship between groups and herds is rather like that between binary and ternary quasigroups, though the parallel is not exact—the different group structures induced by different choices of basepoint in a herd are all isomorphic, whereas the different binary quasigroup structures on a ternary quasigroup may be completely unrelated.)

The same relationship holds between topological groups and topological herds; thus the 'closed subgroup theorem' for topological herds would follow immediately from the theorem for topological groups, if the latter were valid. However, in the localic case the theorem for herds is a genuine extension of the theorem for groups. Recall that in [5, §5], an inverse limit construction was described for producing arbitrarily large localic $\mathbb{T}$-algebras (for any finitary algebraic theory $\mathbb{T}$) whose only points are (the values of) the pseudo-constants of $\mathbb{T}$; since the theory of herds has no pseudo-constants (its only unary operation is the identity), the construction produces nontrivial localic herds which have no points, and therefore cannot carry a localic group structure. Moreover, since the free herd on any set embeds as a subherd of the free group on the same set, the pointless herds produced by the construction all occur as subherds of localic groups, which cannot be represented as cosets of localic subgroups.

## 2. Pregroupoids and herdoids

As originally defined by Kock [11], a *pregroupoid* in a category with finite limits consists of an object $G$ equipped with a morphism $\beta: G \to B$ ($B$ is called the *base* of the pregroupoid) and a quaternary relation $\Lambda \rightarrowtail G^4$, satisfying the following conditions (which we write in set-theoretic notation):

(i) 'Book-keeping axioms': $\Lambda(x,y,z,u)$ implies $\beta(y)=\beta(z)$ and $\beta(x)=\beta(u)$.

(ii) 'The rule of three': Given any three of the four elements $x$, $y$, $z$, $u$ satisfying the appropriate book-keeping axiom, there is a unique choice of the fourth element so that $\Lambda(x,y,z,u)$ holds.

(iii) The binary relations $\sim_h$ on $G \times G$, and $\sim_v$ on $G \times_B G$, defined by

$$(x,y) \sim_h (u,z) \quad \Leftrightarrow \quad \Lambda(x,y,z,u)$$

and

$$(x,u) \sim_v (y,z) \quad \Leftrightarrow \quad \Lambda(x,y,z,u)$$

are equivalence relations.

(N.b.: we have permuted the variables in Kock's original relation $\Lambda$ (by interchanging the first two), for reasons which will become clear shortly.) Kock represents the validity of the relation $\Lambda(x,y,z,u)$ by a parallelogram

$$\begin{tikzcd} u \arrow[r, dash] \arrow[d, equal] & z \arrow[d, equal] \\ x \arrow[r, dash] & y \end{tikzcd}$$

where the double bonds indicate that the pairs of elements which they connect must lie in the same fibre of $\beta$.

There is some redundancy in these axioms. The symmetry of the relations $\sim_h$ and $\sim_v$ means that $\Lambda$ is invariant under the action of the Klein four-group on $G^4$ (i.e. $\Lambda(x,y,z,u) \Leftrightarrow \Lambda(u,z,y,x) \Leftrightarrow \Lambda(y,x,u,z)$), and so any one of the four instances of the 'rule of three' implies the other three. Because of this, it seems sensible to reformulate the axioms in terms of the partial ternary operation $\mu$ whose existence is guaranteed by one of these instances, that is,

$$\mu(x,y,z) = u \quad \Leftrightarrow \quad \Lambda(x,y,z,u).$$

If we do so, they take on a more familiar appearance: they become

(0) $\mu(x,y,z)$ is defined iff $\beta(y)=\beta(z)$ (i.e., $\mu: G\times G\times_B G\to G$).

(1) $\beta\mu(x,y,z)=\beta(x)$ if $\mu(x,y,z)$ is defined.

(2) $\mu(x,x,y)=y$ if $\mu(x,x,y)$ is defined, and $\mu(x,y,y)=x$.

(3) $\mu(x,y,\mu(z,u,v))=\mu(\mu(x,y,z),u,v)$ if either side is defined (note that, by (0) and (1), one side is defined iff the other is).

Here axioms (0) and (1) are the book-keeping axioms and the 'rule of three'; (2) expresses the reflexivity of the relations $\sim_h$ and $\sim_v$, and the associative law (3) (in the presence of (2)) takes care of both symmetry and transitivity of these relations. Thus we see that a pregroupoid is simply 'a herd with book-keeping conditions'; in particular, if $B$ is the terminal object 1 (so that the book-keeping conditions become vacuous), the above axioms reduce precisely to those for a herd. So the relationship between pregroupoids and herds is parallel to that between groupoids and groups.

On the other hand, the relationship between groupoids and pregroupoids does not parallel that between groups and herds: we have a trapezium, not a parallelogram. In [11], Kock observes that, given a groupoid with set of objects $B$, we may obtain a pregroupoid by choosing a point $*\in B$ and taking $G$ to be the set of arrows in the groupoid with domain $*$, with $\beta(x)$ taken to be the codomain of $x$ and $\mu(x,y,z)=xy^{-1}z$. But this process involves an arbitrary choice of basepoint $*$ (so it is not functorial) and it loses information about the part of the groupoid, if any, lying outside the connected component of $*$ (so it is not faithful). However, there is an obvious way to remedy this defect, which was adopted by Kock in the subsequent paper [10]: namely, to remove the asymmetry present in the book-keeping axioms by introducing a second base object.

In [10], Kock retained the name 'pregroupoid' for this more general notion (which he again axiomatized in terms of a quaternary relation rather than a partial ternary operation). We prefer to introduce a new name: we define a *herdoid* (with bases $A$ and $B$) to be a set $G$ equipped with two projections $\alpha: G\to A$, $\beta: G\to B$ and a partial ternary operation $\mu$, satisfying

(0) $\mu(x,y,z)$ is defined iff $\alpha(x)=\alpha(y)$ and $\beta(y)=\beta(z)$ (i.e., $\mu: G\times_A G\times_B G\to G$).

(1) $\alpha\mu(x,y,z)=\alpha(z)$ and $\beta\mu(x,y,z)=\beta(x)$ if $\mu(x,y,z)$ is defined.

(2) $\mu(x,x,y)=y$ if $\mu(x,x,y)$ is defined, and $\mu(x,y,y)=x$ if $\mu(x,y,y)$ is defined.

(3) $\mu(x,y,\mu(z,u,v))=\mu(\mu(x,y,z),u,v)$ if either side is defined.

The notion of herdoid is 'essentially algebraic', and so makes sense in any category with finite limits. Clearly, a pregroupoid (in the original sense) is just a herdoid for which $A = 1$. On the other hand, any groupoid becomes a herdoid if we take both $A$ and $B$ to be the set of objects, $G$ to be the set of arrows, and $\alpha$ and $\beta$ to be the domain and codomain maps. Conversely, given a herdoid for which $A$ and $B$ happen to coincide, any choice of a simultaneous splitting for $\alpha$ and $\beta$ (if such a thing exists) equips it with a groupoid structure. Thus we have completed the parallelogram: $\mu$(herd, group, groupoid) = herdoid.

Incidentally, it is worth remarking that the concept of herdoid provides a simplification of (part of) the proof of the main result of [9]. If we have a natural Mal'cev operation on the objects of a category $\mathscr{E}$ with finite limits, then every span $(A \leftarrow G \rightarrow B)$ in $\mathscr{E}$ carries a natural herdoid structure (just restrict the Mal'cev operation to $G \times_A G \times_B G \rightarrowtail G \times G \times G$; recall from [9] that a natural Mal'cev operation is necessarily associative). Hence, by the remarks in the previous paragraph, every reflexive graph in $\mathscr{E}$ carries a natural groupoid structure.

Before turning to localic herdoids in particular, we note a useful result about herdoids in any category with finite limits.

**Lemma 2.1.** *Let $(G, A, B, \alpha, \beta, \mu)$ be a herdoid in a category $\mathscr{E}$ with finite limits, and let $f: A' \rightarrow A$, $g: B' \rightarrow B$ be any two morphisms in $\mathscr{E}$. Form the pullback*

$$\begin{tikzcd} G' \arrow[r, "h"] \arrow[d, "(\alpha', \beta')"'] & G \arrow[d, "(\alpha, \beta)"] \\ A' \times B' \arrow[r, "f \times g"] & A \times B . \end{tikzcd}$$

*Then $G'$ carries a unique herdoid structure making $(h, f, g)$ a herdoid homomorphism.* □

The proof of Lemma 2.1 is entirely straightforward, and is left to the reader.

We now state the main theorem of this section, which includes both Theorem 1.1 for herds and Theorem 2 of [8] for groupoids as special cases.

**Theorem 2.2.** *Let $(G, A, B, \alpha, \beta, \mu)$ be a localic herdoid, and $(G', A', B', \alpha', \beta', \mu')$ a localic subherdoid (i.e. suppose we are given inclusions $A' \rightarrow A$, $B' \rightarrow B$ and $G' \rightarrow G$ forming a herdoid homomorphism), and suppose further that $\alpha': G' \rightarrow A'$ and $\beta': G' \rightarrow B'$ are open maps. Then the inclusion $G' \rightarrow G$ is fibrewise closed over $A \times B$.*

We note first that, thanks to Lemma 2.1, we may immediately reduce to the case where $A' \rightarrow A$ and $B' \rightarrow B$ are identity maps; for the fibrewise closure of $G'$ in $G$ over $A \times B$ coincides with its fibrewise closure in the pullback of $G$ over $A' \times B'$. The proof thereafter falls into two parts, which exactly parallel those we have given before: we must show that the fibrewise closure (in a suitable sense) of $G'$ in $G$ is a subherdoid of $G$, and then prove an analogue of Lemma 1.3 to cover the case where the inclusion is fibrewise dense over $A$ and over $B$ (so that, by [8, Lemma 1.11(ii)], we may additionally assume that $\alpha$ and $\beta$ are open). We deal with the second of these first; we give the lemma in its greatest generality, which is considerably more than we need.

**Lemma 2.3.** *Let $(G, A, B, \alpha, \beta, \mu)$ be a localic herdoid such that $\alpha$ and $\beta$ are open maps, and let $S$, $T$, $U$ be three sublocales of $G$ such that $S$ is fibrewise dense over $B$, $U$ is fibrewise dense over $A$, and we can write $T = T_1 \cap T_2$ where $T_1$ and $T_2$ are fibrewise dense over $A$ and $B$ respectively. Then:*

(i) *The inclusion $S \times_A T \times_B U \to G \times_A G \times_B G$ is fibrewise dense over $\mu : G \times_A G \times_B G \to G$.*

(ii) *The composite $S \times_A T \times_B U \to G \times_A G \times_B G \xrightarrow{\mu} G$ is epimorphic.*

**Proof.** As usual, the second assertion follows from the first, since $\mu$ is (split) epimorphic, and we prove the first by regarding $S \times_A T \times_B U$ as the intersection of $S \times_A T_1 \times_B G$ and $G \times_A T_2 \times_B U$. First we claim that $T_1 \times_A S \to G \times_A G$ is fibrewise dense over $\beta\pi_2 : G \times_A G \to B$; this is because it is the intersection of $T_1 \times_A G$ and $G \times_A S$, and we have diagrams

$$\begin{tikzcd}
T_1 \times_A G \arrow[r] \arrow[d] & G \times_A G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & G \arrow[d, "\alpha"] \\
T_1 \arrow[r] & G \arrow[r, "\alpha"] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
G \times_A S \arrow[r] \arrow[d] & G \times_A G \arrow[d, "\pi_2"] \arrow[dr, "\beta\pi_2"] & \\
S \arrow[r] & G \arrow[r, "\beta"'] & B
\end{tikzcd}$$

in which the squares are pullbacks. Applying [8, Proposition 2.7], to the first of these, we deduce that $T_1 \times_A G \to G \times_A G$ is fibrewise dense over $\pi_2$ (and hence over $\beta\pi_2$, by [8, Lemma 2.3]). Applying [8, Lemma 1.9], to the second (and noting that $\pi_2 : G \times_A G \to G$ is open, because it is a pullback of $\alpha$), we deduce that $G \times_A S \to G \times_A G$ is fibrewise dense over $\beta\pi_2$.

So $T_1 \times_A S \to G \times_A G$ is fibrewise dense over $\beta\pi_2$; hence, by [8, Proposition 2.7] again, $T_1 \times_A S \times_B G \to G \times_A G \times_B G$ is fibrewise dense over $\pi_3 : G \times_A G \times_B G \to G$. But we have a diagram

$$\begin{tikzcd}
T_1 \times_A S \times_B G \arrow[rr, "(\pi_2, \pi_1, \mu')"] \arrow[d] & & S \times_A T_1 \times_B G \arrow[d] \\
G \times_A G \times_B G \arrow[rr, "(\pi_2, \pi_1, \mu)"] \arrow[dr, "\pi_3"'] & & G \times_A G \times_B G \arrow[dl, "\mu"] \\
& G &
\end{tikzcd}$$

in which the horizontal maps are isomorphisms ($\mu'$ denotes the restriction of $\mu$); so $S \times_A T_1 \times_B G \to G \times_A G \times_B G$ is fibrewise dense over $\mu$. Similarly, $G \times_A T_2 \times_B U \to G \times_A G \times_B G$ is fibrewise dense over $\mu$; so the result follows by [8, Lemma 1.2]. $\square$

The other ingredient needed for the proof of Theorem 2.2 is the following:

**Lemma 2.4.** *Let $(G, A, B, \alpha, \beta, \mu)$ be a localic herdoid, and let $H$ be a sublocale of $G$ which is a subherdoid (with the same bases $A$ and $B$) and such that the restrictions of $\alpha$ and $\beta$ to $H$ are open maps. Then there is a subherdoid $\bar{H}$ of $G$, containing $H$, such that the inclusion $H \to \bar{H}$ is fibrewise dense over both $A$ and $B$, and $\bar{H} \to G$ is fibrewise closed over $A \times B$.*

**Proof.** As in the proof of Theorem 2 of [8], we do not define $\bar{H}$ to be the fibrewise closure of $H$ over $A \times B$, but rather the intersection of its fibrewise closures over $A$ and over $B$; the reason for this is that we cannot prove that the former yields a subherdoid of $G$ unless we make the (unpleasantly restrictive) assumption that $H \to G \to A \times B$ is an open map. (N.b.: in the argument which follows, we shall fail to distinguish notationally between the morphisms $\mu$, $\alpha$ and $\beta$ and their restrictions to either $H$ or $\bar{H}$.)

It is immediate that $\bar{H}$ has the fibrewise properties stated in the lemma, so we have only to prove that it is a subherdoid, i.e. that $\bar{H} \times_A \bar{H} \times_B \bar{H} \to G \times_A G \times_B G \xrightarrow{\mu} G$ factors through $\bar{H} \to G$. To do this, it suffices by the functoriality of fibrewise closure [8, Lemma 1.6] to show that the inclusion $H \times_A H \times_B H \to \bar{H} \times_A \bar{H} \times_B \bar{H}$ is fibrewise dense over both $\alpha\mu = \alpha\pi_3$ and $\beta\mu = \beta\pi_1$. We shall do the second of these; the first is similar.

First, $H \times_A H \times_B H \to \bar{H} \times_A H \times_B H$ is fibrewise dense over $\beta\pi_1$ by [8, Lemma 1.9], because it is a pullback in **Loc**/$B$ of the inclusion $H \to \bar{H}$ along the (open) projection $\pi_1 : \bar{H} \times_A H \times_B H \to \bar{H}$. To show that $\bar{H} \times_A H \times_B H \to \bar{H} \times_A \bar{H} \times_B \bar{H}$ is fibrewise dense over $\pi_1$ (and hence over $\beta\pi_1$), it suffices by [8, Proposition 2.7], to show that $H \times_B H \to \bar{H} \times_B \bar{H}$ is fibrewise dense over $\alpha\pi_1$, since we have pullbacks

$$\begin{tikzcd}
\bar{H} \times_A H \times_B H \arrow[r] \arrow[d, "(\pi_2, \pi_3)"] & \bar{H} \times_A \bar{H} \times_B \bar{H} \arrow[r, "\pi_1"] \arrow[d, "(\pi_2, \pi_3)"] & {} \arrow[d, "\alpha"] \\
H \times_B H \arrow[r] & \bar{H} \times_B \bar{H} \arrow[r, "\alpha\pi_1"] & A.
\end{tikzcd}$$

But this may be proved by another application of the same argument: we factor the inclusion as $H\times_B H\to \bar{H}\times_B H\to \bar{H}\times_B \bar{H}$, where the first factor is fibrewise dense over $\alpha\pi_1$ using [8, Lemma 1.9], and the second is fibrewise dense over $\pi_1$ using [8, Proposition 2.7]. $\square$

The proof of Theorem 2.2 is now a straightforward matter of putting together the three lemmas of this section. First we use Lemma 2.1, as already indicated, to reduce to the case where $A'\to A$ and $B'\to B$ are identity maps. (In fact this step is not strictly necessary; with a little extra effort, Lemma 2.4 could have been formulated in such a way as not to require this hypothesis.) Then Lemma 2.4 allows us to replace $G$ by the closure of $G'$, and so reduce to the case where $G'\to G$ is fibrewise dense over both $A$ and $B$. Finally, putting $S=T=U=G'$ in Lemma 2.3 yields the result.

It should by now be clear, as suggested in the Introduction, that the same methods would yield a proof of the corresponding result for localic 'ternary quasigroupoids', if one knew how to formulate the latter notion. One difficulty associated with this is the problem of deciding what form the 'book-keeping axioms' should take for a ternary quasigroupoid; in the absence of any naturally occurring examples of such structures (which contrasts with the fact that localic herdoids do arise naturally in connection with fibre bundles [10, 11], and localic quasigroups—though not much studied as yet—seem likely to be at least as numerous and interesting as topological quasigroups), there seems to be little point in speculating about this.

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